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Upper Bounds for Online Ramsey Games in Random Graphs

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Consider the following one-player game. Starting with the empty graph on n vertices, in every step a new edge is drawn uniformly at random and inserted into the current graph. This edge has to be coloured immediately with one of r available colours. The player's goal is to avoid creating a monochromatic copy of some fixed graph F for as long as possible. We prove an upper bound on the typical duration of this game if F is from a large class of graphs including cliques and cycles of arbitrary size. Together with lower bounds published elsewhere, explicit threshold functions follow.

1. Introduction

Consider the following one-player game. The board is a graph with n vertices, which initially contains no edges. The edges are presented to the player, henceforth called Painter, one by one in an order chosen uniformly at random among all permutations of the underlying complete graph. Painter must assign one of r available colours to each edge immediately. Her objective is to colour as many edges as possible without creating a monochromatic copy of some fixed graph F. The game ends as soon as the first monochromatic copy of F is closed. We refer to this as the online F-avoidance game with r colours, and call the number of properly coloured edges its *duration*. This game was introduced by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali [1], who showed that the duration of the triangle-avoidance game with two colours is determined by a threshold that is substantially different from that of the offline setting: if a random graph with $cn^{3/2}$ edges is revealed to Painter all at once, she can asymptotically almost surely (a.a.s.) find a colouring without any monochromatic triangle, provided $c > 0$ is sufficiently small [2]. In contrast, the online game a.a.s. ends after roughly $n^{4/3}$ edges.

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‡ A preliminary version of this paper appeared in the proceedings of the 19th International Workshop on Randomized Techniques in Computation (RANDOM'05) [4].

We say that $N_0 = N_0(F, r, n)$ is a threshold for the game if there exists a strategy such that Painter a.a.s. survives with this strategy for any $N \ll N_0$ edges, and if, moreover, Painter a.a.s. loses the game within any $N \gg N_0$ edges, regardless of her strategy.

In [5] it was shown that, for every graph F and every integer $r \geqslant 1$, a threshold for the online F-avoidance game with r colours exists. Moreover, the following lower bound on this threshold was established. For any non-empty graph F and every integer $r \geqslant 1$, let

$$\overline{m}_2^r(F) := \begin{cases} \max\limits_{H \subseteq F} \dfrac{e_H}{v_H} & \text{if } r = 1, \\ \max\limits_{H \subseteq F} \dfrac{e_H}{v_H - 2 + 1/\overline{m}_2^{r-1}(F)} & \text{if } r \geqslant 2. \end{cases} \tag{1.1}$$

Theorem 1.1 ([5]). *Let F be a graph that is not a forest, and let $r \geqslant 1$. Then the online F-avoidance edge-colouring game with r colours has a threshold $N_0(F, r, n)$ that satisfies*

$$N_0(F, r, n) \geqslant n^{2-1/\overline{m}_2^r(F)}.$$

Note that the order of magnitude of this lower bound depends on the number of colours r, in contrast to the well-known offline threshold found by Rödl and Ruciński [6, 7], which is determined by

$$m_2(F) := \max_{H \subseteq F} \frac{e_H - 1}{v_H - 2}. \tag{1.2}$$

It is not difficult to verify (see [5]) that $\overline{m}_2^r(F)$ satisfies

$$\overline{m}_2^1(F) < \overline{m}_2^2(F) < \cdots < \overline{m}_2^r(F) < \cdots < m_2(F)$$

and

$$\lim_{r \to \infty} \overline{m}_2^r(F) = m_2(F).$$

It follows that the threshold of the online game approaches the offline threshold as more colours are available to the player. In this paper we prove a matching upper bound for the game with two colours and graphs F from a large class of graphs including cliques and cycles of arbitrary size.

Theorem 1.2 (Main result). *Let F be a graph that is not a forest, which has a subgraph $F_- \subset F$ with $e_F - 1$ edges satisfying*

$$m_2(F_-) \leqslant \overline{m}_2^2(F). \tag{1.3}$$

Then the threshold for the online F-avoidance edge-colouring game with two colours is

$$N_0(F, 2, n) = n^{2-1/\overline{m}_2^2(F)}.$$

We believe that a similar result is true for all $r \geqslant 2$, as was shown in [3] for the analogous vertex-colouring problem. In Section 4 we briefly discuss this as an open problem.

We close this section by stating the resulting threshold functions for clique- and cycle-avoidance games explicitly.

Corollary 1.3 (Clique-avoidance games). *For all $\ell \geqslant 2$, the threshold for the online K_ℓ-avoidance edge-colouring game with two colours is*

$$N_0(\ell, 2, n) = n^{\left(2-\frac{2}{\ell+1}\right)\left(1-\binom{\ell}{2}^{-2}\right)}.$$

Corollary 1.4 (Cycle-avoidance games). *For all $\ell \geqslant 3$, the threshold for the online C_ℓ-avoidance edge-colouring game with two colours is*

$$N_0(\ell, 2, n) = n^{1+1/\ell}.$$

1.1. Organization of this paper

We conclude this Introduction by explaining our notation and summarizing some auxiliary results. Section 2 is devoted to the proof of Theorem 1.2. Our argument in Section 2 relies on two combinatorial statements which are proved in Section 3. We conclude the paper by briefly commenting on open questions in Section 4.

1.2. Preliminaries and notation

We consider the random graph process $(G(n,N))_{0\leqslant N\leqslant\binom{n}{2}}$, where the edges appear uniformly at random one after the other, *i.e.*, in one of $\binom{n}{2}!$ possible permutations. It is easily seen that $G(n,N)$ is uniformly distributed over all graphs on n vertices with exactly N edges. We denote a graph chosen uniformly at random from all graphs on n vertices with exactly $m = m(n)$ edges by $G_{n,m}$. In the binomial model, $G_{n,p}$ denotes a random graph on n labelled vertices in which each edge is present with probability $p = p(n)$ independently of all other edges. Since both models are equivalent in terms of asymptotic properties if $m \asymp pn^2$ and p is sufficiently large, we sometimes switch from one to the other to simplify the presentation.

The following theorem from [7] is a counting version of the threshold result for the offline case that was mentioned above.

Theorem 1.5 ([7]). *Let $r \geqslant 2$ and F be a non-empty graph. Then there exist positive constants $C = C(F,r)$ and $a = a(F,r)$ such that, for*

$$p(n) \geqslant Cn^{-1/m_2(F)},$$

where $m_2(F)$ is defined as in (1.2)*, the random graph $G_{n,p}$ a.a.s. satisfies the property that in every r-edge-colouring there are at least $an^{v_F}p^{e_F}$ monochromatic copies of F.*

All graphs are simple and undirected. The number of vertices of a graph G is denoted by v_G or $v(G)$, and similarly the number of edges by e_G or $e(G)$. We denote a clique on ℓ vertices by K_ℓ and a cycle on ℓ vertices by C_ℓ.

The standard density measure for graphs is $d(G) := e_G/v_G$, which is exactly half of the average degree. Besides $d(G)$, we also use the so-called 2-density $d_2(G) := (e_G - 1)/(v_G - 2)$. For the sake of completeness, we also define $d_2(K_2) := 1/2$ and $d(G) = d_2(G) := 0$ if G is empty. For a given density function d_i, we let $m_i(G) := \max_{H\subseteq G} d_i(H)$. We say that G is *balanced with respect to d_i* if $m_i(G) = d_i(G)$. We simply write *balanced* for balancedness w.r.t. d, and 2-*balanced* for balancedness w.r.t. d_2.

As we will only consider the game with two colours throughout this paper, we abbreviate $\overline{m}_2^2$ by $\overline{m}_2$. For non-empty graphs F and G, we define

$$\overline{d}_2(F,G) := \frac{e_G}{v_G - 2 + 1/m(F)},$$

and set $\overline{d}_2(F,G) := 0$ if F or G is empty. Note that due to (1.1) we have

$$\overline{m}_2(F) = \max_{H \subseteq F} \overline{d}_2(F,H).$$

For further reference, we state the following elementary observation.

Proposition 1.6. *For a, c, $C \in \mathbb{R}$ and b, $d > 0$, we have*

$$\frac{a}{b} \leqslant C \ \wedge \ \frac{c}{d} \leqslant C \quad \Longrightarrow \quad \frac{a+c}{b+d} \leqslant C \quad \text{and} \quad \frac{a}{b} \geqslant C \ \wedge \ \frac{c}{d} \geqslant C \quad \Longrightarrow \quad \frac{a+c}{b+d} \geqslant C,$$

and similarly, if $b > d$,

$$\frac{a}{b} \leqslant C \ \wedge \ \frac{c}{d} \geqslant C \quad \Longrightarrow \quad \frac{a-c}{b-d} \leqslant C \quad \text{and} \quad \frac{a}{b} \geqslant C \ \wedge \ \frac{c}{d} \leqslant C \quad \Longrightarrow \quad \frac{a-c}{b-d} \geqslant C.$$

2. Proof of Theorem 1.2

In this section we prove Theorem 1.2. In light of Theorem 1.1 it suffices to prove that $n^{2-1/\overline{m}_2(F)}$ is an upper bound on the threshold of the online F-avoidance game. That is, we need to show that for any $N \gg n^{2-1/\overline{m}_2(F)}$ Painter will a.a.s. close a monochromatic copy of F within N moves, regardless of her strategy. In order to do so we relax the online F-avoidance game to an offline two-round game, where we grant a mercy period of N_1 edges to Painter. She may wait until the end of this phase with the colouring of those edges. Then another $N_2 = N - N_1$ random edges are simultaneously added, and Painter must colour them. We argue that, regardless of her strategy, she will a.a.s. create many 'threats' in the first round which force her to create a monochromatic copy of F in the second round. Let R and B denote the subgraphs of $G(n, N_1)$ spanned by the red and blue edges respectively, and let the *base graph* of R, denoted by $\mathrm{Base}(R)$, be the set of all vertex-pairs that, joined by an edge, would complete a subgraph isomorphic to F in R. $\mathrm{Base}(B)$ is defined analogously. Clearly, if an edge from $\mathrm{Base}(R)$ (or $\mathrm{Base}(B)$) is added to the graph, it has to be coloured blue (resp. red). The 'threats' are copies of F in $\mathrm{Base}(R)$ or $\mathrm{Base}(B)$. A sufficiently large number of threats in $\mathrm{Base}(R)$ will ensure that Painter a.a.s. creates a blue copy of F in the second round (and *vice versa*), which ends the game.

Let us give some intuition why this idea yields an upper bound of $n^{2-1/\overline{m}_2(F)}$ on the duration of the game. Assume for simplicity that $\overline{m}_2(F) = \overline{d}_2(F,F)$ and $m(F) = d(F)$, such that

$$\overline{m}_2(F) = \frac{e_F}{v_F - 2 + v_F/e_F} = \frac{e_F^2}{e_F(v_F - 2) + v_F}. \tag{2.1}$$

For a fixed $N = N(n)$, choose $N_1 = N_2 := N/2$ as the duration of both rounds. Letting $p := N/n^2$, we switch to the $G_{n,p}$ model to simplify presentation. Note that the edges in $\mathrm{Base}(R)$ are induced by red copies of subgraphs $F_- \subset F$ with $e_F - 1$ edges. After the first round, the expected number of copies of such graphs in $G(n, N_1)$ is $\Theta(n^{v_F} p^{e_F - 1})$, and with Theorem 1.5 we can find

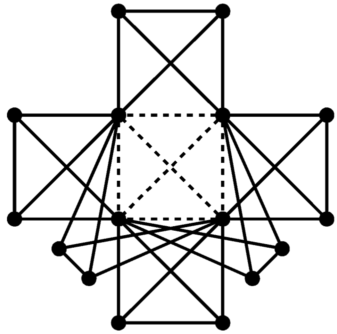

Figure 1. The unique member of the class $\mathcal{F}^*(K_4, K_4)$. Removing the dashed inner edges yields the unique member of $\mathcal{F}^*_-(K_4, K_4)$.

asymptotically the same number of monochromatic (w.l.o.g. red) copies of F_-. If most of these induce different edges in Base(R), we have about $\Theta(n^{v_F} p^{e_F - 1})$ such edges. We define

$$p_B := n^{v_F - 2} p^{e_F - 1}$$

as the expected edge density of Base(R). If the edges in Base(R) are distributed as in a random graph G_{n,p_B}, they form $\Theta(n^{v_F} p_B^{e_F})$ copies of F. As explained above, these copies form 'threats' for the second round. The expected number of such threats being hit in the second round is of order

$$(n^{v_F} p_B^{e_F}) p^{e_F} = n^{v_F} (n^{v_F - 2} p^{e_F})^{e_F} = n^{v_F + e_F(v_F - 2)} p^{e_F^2},$$

so we expect that Painter does not survive the second round if

$$p \gg n^{-(v_F + e_F(v_F - 2))/e_F^2} \overset{(2.1)}{=} n^{-1/\overline{m}_2(F)}.$$

Clearly, this two-round approach can only work if the first round creates many edges in one of the base graphs Base(R) and Base(B). Theorem 1.5 only guarantees this if $m_2(F_-) \leqslant \overline{m}_2(F)$ for an $F_- \subset F$. This explains why condition (1.3) is needed in our framework.

There are two main technical issues with this approach. Firstly, it is not clear that the monochromatic copies of F_- induce many *distinct* edges in the base graph. Secondly, we need to deal with the fact that edges in Base(R) are not mutually independent. We overcome these difficulties by refining our approach as follows. Instead of 'building' the threats by first looking for edges in Base(R) and then constructing copies of F from those, we look for red copies of graphs that directly induce *complete copies* of F in Base(R).

In order to make these ideas precise, we define two graph classes which will be important in the proof. We start with $\mathcal{F}^*(H_1, H_2)$, the class of all graphs obtained by embedding all edges of an inner copy of H_1 into edge-disjoint outer copies of H_2. Figure 1 shows the unique member of $\mathcal{F}^*(K_4, K_4)$ as an example.

Definition 1. For non-empty graphs H_1 and H_2, let

$$\mathcal{F}^*(H_1, H_2) := \Big\{ F^* = \Big(V \,\dot\cup \dot{\bigcup}_{f \in E} U(f),\; E \,\dot\cup \dot{\bigcup}_{f \in E} D(f) \Big) : \\ (V, E) \cong H_1 \quad \wedge \quad \big(f \,\dot\cup\, U(f), \{f\} \,\dot\cup\, D(f) \big) \cong H_2, \quad \forall f \in E \Big\}.$$

The sets V and E form the inner copy of H_1. Every edge $f \in E$ together with $U(f)$ and $D(f)$ forms an outer copy of H_2. Hence, $|U(f)| = v(H_2) - 2$ and $|D(f)| = e(H_2) - 1$. For a given graph $F^* \in \mathcal{F}^*(H_1, H_2)$, we write $U := \dot{\bigcup}_{f \in E} U(f)$ and $D := \dot{\bigcup}_{f \in E} D(f)$, so that $F^* = (V \mathbin{\dot\cup} U, E \mathbin{\dot\cup} D)$.

To simplify notation, we abbreviate $\mathcal{F}^*(F, F)$ by $\mathcal{F}^*$. The next lemma relates the maximum density $\overline{m}_2(F)$ to the graph class $\mathcal{F}^*$.

Lemma 2.1. *Let F be a graph that is not a forest. Then*

$$\overline{m}_2(F) = \max_{F^* \in \mathcal{F}^*} m(F^*).$$

The proof is given in Section 3. For now we just give some intuition why Lemma 2.1 should hold. It is plausible that for symmetry reasons, maximal density among all subgraphs T of all $F^* \in \mathcal{F}^*$ is attained by a nicely structured subgraph $\widehat{T} \in \mathcal{F}^*(G, H)$ consisting of an inner graph G with each edge covered by a copy of H. Maximizing over all such subgraphs of all graphs F^*, we obtain

$$\begin{aligned} \max_{F^* \in \mathcal{F}^*} m(F^*) \geqslant \max_{\substack{G,H \subseteq F \\ \widehat{T} \in \mathcal{F}^*(G,H)}} \frac{e(\widehat{T})}{v(\widehat{T})} &= \max_{G,H \subseteq F} \frac{e_G e_H}{e_G(v_H - 2) + v_G} \\ &= \max_{H \subseteq F} \frac{e_H}{v_H - 2 + 1/m(F)} = \overline{m}_2(F). \end{aligned}$$

Lemma 2.1 asserts that this inequality is in fact an equality.

Next we define the graph class $F^*_-(H_1, H_2)$, which is obtained by removing the inner copy from graphs $F^* \in \mathcal{F}^*(H_1, H_2)$. Again the definition is illustrated in Figure 1.

Definition 2. For non-empty graphs H_1 and H_2, let

$$\mathcal{F}^*_-(H_1, H_2) := \{F^*_- = (V \mathbin{\dot\cup} U, D) : F^* = (V \mathbin{\dot\cup} U, E \mathbin{\dot\cup} D) \in \mathcal{F}^*(H_1, H_2)\}.$$

As before, we abbreviate $\mathcal{F}^*_-(F, F)$ by $\mathcal{F}^*_-$. The proof of the next lemma is also deferred to Section 3.

Lemma 2.2. *Let F be a graph that is not a forest. If there exists a subgraph $F_- \subset F$ with $e_F - 1$ edges satisfying*

$$m_2(F_-) \leqslant \overline{m}_2(F),$$

then there exists $F^*_- \in \mathcal{F}^*_-$ with

$$m_2(F^*_-) \leqslant \overline{m}_2(F).$$

Clearly, every red copy of a graph $F^*_- \in \mathcal{F}^*_-$ induces its missing inner copy in Base(R). Thus, Lemma 2.2 implies with Theorem 1.5 that for $p \gg n^{-1/\overline{m}_2(F)}$, the condition $m_2(F_-) \leqslant \overline{m}_2(F)$ not only ensures that there are many edges in Base(R), but also that these form many copies of F.

Proof of Theorem 1.2. Fix $N(n) \gg n^{2-1/\overline{m}_2(F)}$. Without loss of generality, we assume that $N \ll n^2$. Consider a fixed strategy for Painter, and let the random variable Y denote the duration of the game when played with this strategy. Fix $F_-^* \in \mathcal{F}_-^*$ as provided by Lemma 2.2. Set $N_1 = N_2 := N/2$ and $p := N_2/\binom{n}{2}$. We will tacitly switch between the models $G_{n,p}$ and $G(n,N)$, exploiting their asymptotic equivalence.

Consider the colouring assigned by Painter to the first N_1 edges. Every monochromatic copy of F_-^* induces a copy of F in the corresponding base graph, and Painter must assign the opposite colour to each edge of that F if it appears in the second round. Hence, she loses if the $N_2 \asymp N$ remaining edges form any such copy of F. Since $G(n,N_1)$ contains only $o(n^2)$ edges, the probability of that event is asymptotically equal to the probability that an independently generated graph G_{n,N_2} or $G_{n,p}$ contains one of those copies of F.

Let the random variable M denote the number of monochromatic copies of F_-^* after the first round. It is determined by the outcome of $(G(n,N))_{0\leqslant N\leqslant N_1}$ and Painter's strategy. For the second round, consider M being fixed. For each monochromatic copy $F_{-i}^* \subseteq G(n,N_1)$, $i = 1,\dots,M$, let F^i denote its induced inner copy of F. Let the random variable Z_i denote the event that F^i is in $G_{n,p}$, and let $Z := \sum_{i=1}^M Z_i$. Note that $Z_i \equiv Z_j$ if F_{-i}^* and F_{-j}^* induce the same copy $F^i = F^j$ in $\mathrm{Base}(R)$ or $\mathrm{Base}(B)$, and that therefore the same threat may be considered multiple times in the definition of Z.

We have

$$\mathbb{E}[Z] = Mp^{e_F}$$

and

$$\begin{aligned}
\mathrm{Var}[Z] &= \mathbb{E}[Z^2] - \mathbb{E}[Z]^2 \\
&= \sum_{i,j=1}^{M} \bigl(\mathbb{E}[Z_iZ_j] - \mathbb{E}[Z_i]\mathbb{E}[Z_j]\bigr) \\
&= \sum_{\substack{G\subseteq F\\ e_G\geqslant 1}} \sum_{\substack{i,j=1\\ F^i\cap F^j\cong G}}^{M} \bigl(p^{2e_F-e_G} - p^{2e_F}\bigr) \\
&\leqslant \sum_{\substack{G\subseteq F\\ e_G\geqslant 1}} M_G p^{2e_F-e_G},
\end{aligned}$$

where the random variable M_G denotes the number of pairs of monochromatic copies of F_-^* whose induced inner copies intersect in a copy of G. Like M, consider M_G being fixed after the first round.

In the remainder of this proof, we let the random variable X_G denote the number of copies of a fixed graph G in $G_{n,p}$. Note that $\mathbb{E}[X_G] \asymp n^{v_G}p^{e_G}$. Lemma 2.1 implies with $p \gg n^{-1/\overline{m}_2(F)}$ that all subgraphs T of all $F^* \in \mathcal{F}^*$ satisfy

$$\mathbb{E}[X_T] = \omega(1). \tag{2.2}$$

Moreover, Theorem 1.5 yields that a.a.s. we have $M = \Omega(\mathbb{E}[X_{F_-^*}])$, from which we obtain that

$$\mathbb{E}[Z] = \Omega(\mathbb{E}[X_{F_-^*}])p^{e_F} = \Omega(\mathbb{E}[X_{F^*}]) \overset{(2.2)}{=} \omega(1) \tag{2.3}$$

a.a.s.

Clearly, we can bound M_G by the number of pairs of copies of F_-^* whose induced inner copies intersect in a copy of G, regardless of their colouring. Moreover, the number of ways in which two copies of F_-^* may overlap is finite. Let $F_-^* \cup_G F_-^*$ denote a fixed graph obtained as the union of two copies of F_-^* whose (missing) inner copies intersect in a copy of G, and let $T \subseteq F_-^* \cup_G F_-^*$ denote the intersection of the two copies of F_-^*. Note that all vertices of G are in both copies of F_-^* and thus in T. Hence, we may define T^+ as the graph in which the edges of G are added to T. We obtain

$$\mathbb{E}[X_{F_-^* \cup_G F_-^*}] = \Theta\left(\frac{\mathbb{E}[X_{F_-^*}]^2}{\mathbb{E}[X_T]}\right) = \Theta\left(\frac{\mathbb{E}[X_{F_-^*}]^2 p^{e_G}}{\mathbb{E}[X_{T^+}]}\right) \overset{(2.2)}{=} o\big(\mathbb{E}[X_{F_-^*}]^2 p^{e_G}\big).$$

Since the number of isomorphism classes of graphs of type $F_-^* \cup_G F_-^*$ is bounded by a constant only depending on F, it follows by the first moment method that a.a.s. $M_G = o\big(\mathbb{E}[X_{F_-^*}]^2 p^{e_G}\big)$ for all non-empty $G \subseteq F$. We obtain that a.a.s.

$$\begin{aligned}\mathrm{Var}[Z] &= \sum_{\substack{G\subseteq F\\ e_G \geqslant 1}} o\big(\mathbb{E}[X_{F_-^*}]^2 p^{e_G}\big) p^{2e_F - e_G}\\ &= o(\mathbb{E}[X_{F^*}]^2)\\ &\overset{(2.3)}{=} o(\mathbb{E}[Z]^2).\end{aligned}$$

Now the second moment method yields that $\mathbb{P}[Y \geqslant N] \leqslant \mathbb{P}[Z = 0] = o(1)$, and Theorem 1.2 is proved. $\square$

3. Proofs of Lemmas 2.1 and 2.2

We first prove Lemma 2.2.

Proof of Lemma 2.2. Fix $H \subseteq F$ satisfying $\overline{d}_2(F,H) = \overline{m}_2(F)$. Consider a graph F_- as given by the assumption. If necessary, add isolated vertices to F_- until it has v_F vertices. Clearly, this maintains the property $m_2(F_-) \leqslant \overline{m}_2(F)$. For this F_-, fix any $F_-^* \in \mathcal{F}_-^*$ with every outer copy $\widehat{F}_-(f) := (f \mathbin{\dot\cup} U(f), D(f)), f \in E$ isomorphic to F_- (*cf.* Definition 1). Slightly abusing notation, we let $F = (V,E)$ denote the missing inner copy of F_-^*. For any fixed subgraph $T \subseteq F_-^*$, let

$$G := F[V(T) \cap V]$$

denote the subgraph of F induced by the inner vertices of T. In order to show that $d_2(T) \leqslant \overline{m}_2(F)$, we compare T to a 'nice' graph $\widehat{T}$, where by nice we mean that $\widehat{T} \in \mathcal{F}_-^*(G,H)$. Note that $\widehat{T}$ is not necessarily a subgraph of F_-^*. For the rest of the proof, $\widehat{T}$ denotes a fixed graph from $\mathcal{F}_-^*(G,H)$. The actual choice is irrelevant since all graphs in this family have the same number of edges and vertices. Note that H depends only on F, while G varies with the choice of $T \subseteq F_-^*$. This construction is illustrated in Figure 2 for the case $F = H = K_4$.

Throughout this proof, let $E_0 := E(G)$ and $V_0 := V(G) = V(T) \cap V$. For every $f \in E_0$, let $J_f := (\widehat{F}_-(f) \cap T) \cup f$ denote the intersection of T with the outer copy $\widehat{F}_-(f)$ complemented with the edge f. For $f \in E \setminus E_0$, define similarly $J_f := (\widehat{F}_-(f) \cap T)$. With these definitions, we

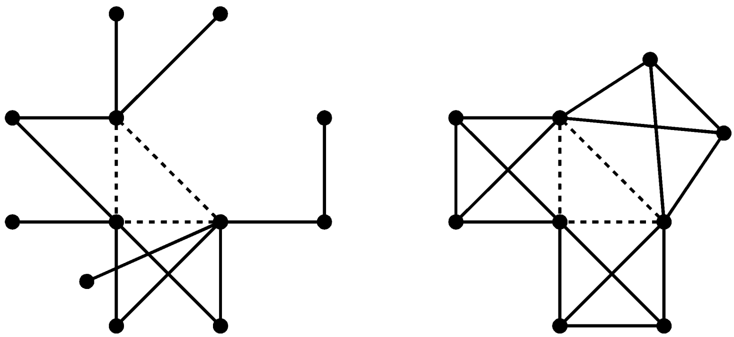

Figure 2. A subgraph T of $F_-^* \in \mathcal{F}_-^*(K_4, K_4)$ (left), and its corresponding $\widehat{T} \in \mathcal{F}_-^*(K_3, K_4)$ (right). The dashed edges form $G = K_3$, but do not belong to either graph.

obtain that

$$e(T) = \sum_{f \in E_0} (e(J_f) - 1) + \sum_{f \in E \setminus E_0} e(J_f) \tag{3.1}$$

and

$$\begin{aligned} v(T) &= v_G + \sum_{f \in E_0} (v(J_f) - 2) + \sum_{\substack{f \in E: \\ |f \cap V_0| = 1}} (v(J_f) - 1) + \sum_{\substack{f \in E: \\ |f \cap V_0| = 0}} v(J_f) \\ &\geqslant v_G + \sum_{f \in E_0} (v(J_f) - 2) + \sum_{\substack{f \in E \setminus E_0: \\ v(J_f) \geqslant 2}} (v(J_f) - 1). \end{aligned} \tag{3.2}$$

With

$$e(\widehat{T}) = e_G(e_H - 1) = \sum_{f \in E_0} (e_H - 1),$$

we obtain, due to (3.1),

$$\begin{aligned} e(T) &= e(\widehat{T}) - \sum_{f \in E_0} (e_H - 1) + \sum_{f \in E_0} (e(J_f) - 1) + \sum_{f \in E \setminus E_0} e(J_f) \\ &= e(\widehat{T}) + \sum_{\substack{f \in E_0: \\ v(J_f) \geqslant v_H}} (e(J_f) - e_H) - \sum_{\substack{f \in E_0: \\ v(J_f) < v_H}} (e_H - e(J_f)) + \sum_{\substack{f \in E \setminus E_0: \\ v(J_f) \geqslant 2}} e(J_f) \\ &\leqslant e(\widehat{T}) + \sum_{\substack{f \in E_0: \\ v(J_f) > v_H}} (e(J_f) - e_H) - \sum_{\substack{f \in E_0: \\ v(J_f) < v_H}} (e_H - e(J_f)) + \sum_{\substack{f \in E \setminus E_0: \\ v(J_f) \geqslant 2}} e(J_f), \end{aligned} \tag{3.3}$$

where in the last line we used that H is a graph with maximal number of edges on v_H vertices.

Similarly, we obtain with

$$v(\widehat{T}) = v_G + e_G(v_H - 2) = v_G + \sum_{f \in E_0} (v_H - 2),$$

from (3.2), that

$$v(T) \geqslant v(\widehat{T}) - \sum_{f\in E_0}(v_H - 2) + \sum_{f\in E_0}(v(J_f) - 2) + \sum_{\substack{f\in E\setminus E_0:\\ v(J_f)\geqslant 2}} (v(J_f) - 1)$$

$$= v(\widehat{T}) + \sum_{\substack{f\in E_0:\\ v(J_f)>v_H}} (v(J_f) - v_H) - \sum_{\substack{f\in E_0:\\ v(J_f)<v_H}} (v_H - v(J_f)) + \sum_{\substack{f\in E\setminus E_0:\\ v(J_f)\geqslant 2}} (v(J_f) - 1). \tag{3.4}$$

Together, (3.3) and (3.4) yield

$$d_2(T) \leqslant \frac{(e(\widehat{T}) - 1) + \sum_{\substack{f\in E_0:\\ v(J_f)>v_H}} (e(J_f) - e_H) - \sum_{\substack{f\in E_0:\\ v(J_f)<v_H}} (e_H - e(J_f)) + \sum_{\substack{f\in E\setminus E_0:\\ v(J_f)\geqslant 2}} e(J_f)}{(v(\widehat{T}) - 2) + \sum_{\substack{f\in E_0:\\ v(J_f)>v_H}} (v(J_f) - v_H) - \sum_{\substack{f\in E_0:\\ v(J_f)<v_H}} (v_H - v(J_f)) + \sum_{\substack{f\in E\setminus E_0:\\ v(J_f)\geqslant 2}}(v(J_f) - 1)}. \tag{3.5}$$

If $e_G \geqslant 1$, we have $v(\widehat{T}) \geqslant v_H \geqslant 3$ and $e(\widehat{T}) \geqslant e_H - 1 \geqslant 2$. It follows by Proposition 1.6 that $d_2(T) \leqslant \overline{m}_2(F)$ if

$$d_2(\widehat{T}) \leqslant \overline{m}_2(F) \quad \text{for all } G \subseteq F \text{ and } \widehat{T} \in \mathcal{F}^*_-(G,H), \tag{3.6}$$

$$\frac{e_J - e_H}{v_J - v_H} \leqslant \overline{m}_2(F) \quad \text{for all } J = J_f \subseteq F \text{ with } v_J > v_H, \tag{3.7}$$

$$\frac{e_H - e_J}{v_H - v_J} \geqslant \overline{m}_2(F) \quad \text{for all } J = J_f \subseteq F \text{ with } 2 \leqslant v_J < v_H, \tag{3.8}$$

$$\frac{e_J}{v_J - 1} \leqslant \overline{m}_2(F) \quad \text{for all } J = J_f \subseteq F \text{ with } v_J \geqslant 2. \tag{3.9}$$

To verify (3.6), observe that

$$d_2(\widehat{T}) = \frac{e_G(e_H - 1) - 1}{e_G(v_H - 2) + v_G - 2} = \frac{e_G e_H - (e_G + 1)}{e_G(v_H - 2 + 1/m(F)) - (e_G/m(F) - (v_G - 2))}$$

is less than $\overline{m}_2(F) = \overline{d}_2(F,H)$ if $e_G/m(F) \leqslant v_G - 2$. Otherwise (3.6) follows by Proposition 1.6 from

$$\frac{e_G + 1}{e_G/m(F) - (v_G - 2)} \geqslant \overline{m}_2(F),$$

which is equivalent to

$$e_G\left(\frac{\overline{m}_2(F)}{m(F)} - 1\right) \leqslant 1 + \overline{m}_2(F)(v_G - 2)$$

and

$$e_G \leqslant \frac{m(F)}{\overline{m}_2(F) - m(F)}\left(1 + \overline{m}_2(F)(v_G - 2) + \frac{\overline{m}_2(F)}{m(F)} - \frac{\overline{m}_2(F)}{m(F)}\right) = \frac{m(F)\overline{m}_2(F)}{\overline{m}_2(F) - m(F)}\big(v_G - 2 + 1/m(F)\big) - 1.$$

Plugging $\overline{m}_2(F) \geqslant \overline{d}_2(F,G) = e_G/(v_G - 2 + 1/m(F))$ into the numerator, one sees that the claim follows if

$$e_G\left(\frac{m(F)}{\overline{m}_2(F) - m(F)} - 1\right) \geqslant 1$$

for all $e_G \geqslant 1$. This is equivalent to $m(F) \geqslant 2\overline{m}_2(F)/3$, and after plugging in $\overline{m}_2(F) = e_H/(v_H - 2 + 1/m(F))$ to

$$m(F) \geqslant \frac{2e_H - 3}{3v_H - 6}.$$

The last inequality is certainly true if $e_H/v_H \geqslant (2e_H - 3)/(3v_H - 6)$ for all graphs H with $e_H \geqslant v_H \geqslant 3$. This is obvious for all $v_H \geqslant 6$ and easily verified by case checking if $v_H < 6$.

Inequality (3.7) follows from

$$\frac{e_J - e_H}{v_J - v_H} = \frac{e_J - e_H}{(v_J - 2 + 1/m(F)) - (v_H - 2 + 1/m(F))} \overset{\text{Prop. 1.6}}{\leqslant} \overline{d}_2(F,H) = \overline{m}_2(F).$$

Similarly, (3.8) follows from

$$\frac{e_H - e_J}{v_H - v_J} = \frac{e_H - e_J}{(v_H - 2 + 1/m(F)) - (v_J - 2 + 1/m(F))} \overset{\text{Prop. 1.6}}{\geqslant} \overline{d}_2(F,H) = \overline{m}_2(F).$$

For (3.9), observe that for all $J \subseteq F$ with $v_J \geqslant 2$,

$$\frac{e_J}{v_J - 1} \leqslant \frac{e_J}{v_J - 2 + 1/m(F)} = \overline{d}_2(F,J) \leqslant \overline{m}_2(F),$$

as $m(F) \geqslant 1$, due to our assumption that F is not a forest.

As explained, it follows from (3.5) with (3.6), (3.7), (3.8), (3.9), and Proposition 1.6 that $d_2(T) \leqslant \overline{m}_2(F)$. This settles the case $e_G \geqslant 1$.

If $e_G = 0$, we assume w.l.o.g. that T is connected. It is easily seen that this implies that G is also connected, and that therefore $v_G \leqslant 1$. If $v_G = 0$, the claim follows directly from the assumption $m_2(F_-) \leqslant \overline{m}_2(F)$. If $v_G = 1$, the claim follows with Proposition 1.6 from

$$d_2(T) \leqslant \frac{\sum_{f\in E} e(J_f) - 1}{\sum_{f\in E}(v(J_f) - 1) - 1},$$

including the two -1 in the first summand and using $m_2(F_-) \leqslant \overline{m}_2(F)$ for this term, and (3.9) for the remaining ones. □

Note that we only used the assumption $m_2(F_-) \leqslant \overline{m}_2(F)$ for the case where G contains no edges.

The framework from the proof of Lemma 2.2 also lends itself to proving Lemma 2.1.

Proof of Lemma 2.1. With the notation of the previous proof, it follows analogously to (3.5) that, for a given subgraph T of an $F^* \in \mathcal{F}^*$,

$$d(T) \leqslant \frac{e(\widehat{T}) + \sum_{\substack{f\in E_0:\\ v(J_f)>v_H}} (e(J_f) - e_H) - \sum_{\substack{f\in E_0:\\ v(J_f)<v_H}} (e_H - e(J_f)) + \sum_{\substack{f\in E\setminus E_0:\\ v(J_f)\geqslant 2}} e(J_f)}{v(\widehat{T}) + \sum_{\substack{f\in E_0:\\ v(J_f)>v_H}} (v(J_f) - v_H) - \sum_{\substack{f\in E_0:\\ v(J_f)<v_H}} (v_H - v(J_f)) + \sum_{\substack{f\in E\setminus E_0:\\ v(J_f)\geqslant 2}} (v(J_f) - 1)},$$

for a $\widehat{T} \in \mathcal{F}^*(G,H)$. Replacing (3.6) by

$$d(\widehat{T}) = \frac{e_G e_H}{e_G(v_H - 2) + v_G} = \frac{e_H}{v_H - 2 + v_G/e_G} \leqslant \frac{e_H}{v_H - 2 + 1/m(F)} = \overline{m}_2(F), \tag{3.10}$$

and using (3.7), (3.8), and (3.9), Proposition 1.6 implies that $d(T) \leqslant \overline{m}_2(F)$, as before.

This proves that $\max_{F^* \in \mathcal{F}^*} m(F^*) \leqslant \overline{m}_2(F)$. The other inequality follows from the fact that any graph from $\mathcal{F}^*(G,H)$ attains equality in (3.10) if $d(G) = m(F)$ and $\overline{d}_2(F,H) = \overline{m}_2(F)$. □

4. Open questions

4.1. General graphs

It is tempting to dismiss the precondition (1.3) as an artifact of our proof strategy and conjecture that every non-forest F has the threshold $n^{2-1/\overline{m}_2^2(F)}$. However, such a conjecture would be wrong – in [5] we gave an example of a graph for which an *ad hoc* strategy yields a better lower bound than guaranteed by Theorem 1.1.

4.2. More colours

We have no proof of any non-trivial upper bound for the game with more than two colours. Nevertheless, we believe that the densities (1.1) determine threshold functions for the game with an arbitrary number r of colours, provided condition (1.3) is satisfied. The intuition behind this conjecture is as follows. As explained in Section 2, if (1.3) holds, the base graph of one colour will have density $p_B = n^{v_F-2}p^{e_F-1}$ after the first round, and a p-fraction of these edges will be hit in the second round. Assuming that these edges are distributed as in a random graph $G_{n,p_B p}$, the second round restricted to these edges is an online game with one colour excluded, which Painter will lose if $p_B p = n^{v_F-2}p^{e_F} \gg n^{-1/\overline{m}_2^{r-1}(F)}$, or, equivalently, if $p \gg n^{-1/\overline{m}_2^{r}(F)}$.

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